UDC 536.248

A NONLINEAR TIME-INDEPENDENT SYSTEM OF DIFFERENTIAL EQUATIONS OF HEAT AND MASS TRANSFER

M. S. Smirnov

Inzhenerno-Fizicheskii Zhurnal, Vol. 9, No. 2, pp. 250–254, 1965

A nonlinear system of differential equations of heat and mass transfer is examined under steady-state conditions. Exact analytical solutions are found to five boundary problems for this system.

Heat and mass transfer processes are governed by the system of nonlinear parabolic equations

$$\frac{\partial u}{\partial \tau} = \nabla [a \nabla u + a \delta \nabla t] + h(u, t),$$

$$\frac{\partial t}{\partial \tau} = \frac{1}{c\gamma} \nabla (\lambda \nabla t) + \varepsilon \frac{\rho}{c} \frac{\partial u}{\partial \tau} + q(u, t) \tag{1}$$

with appropriate initial and boundary conditions. Functions h and q take into account the influence of heat and mass sources and sinks.

The system (1) combines two parabolic equations related to one another by the additional terms containing derivatives with respect to time and the space coordinates. There is definite interest in the solution of (1) for the steady-state case when the process does not depend on time and the partial derivatives with respect to time may be omitted.

The steady-state solution is also of interest because the influence of the dependence of the heat and mass transfer characteristics on temperature t and mass transfer potential u is particularly noticeable not at the start of the process, but later, when the process approaches the steady-state condition. In other words, the nonconstancy of the coefficients of (1) will have its greatest effect from a certain time onwards.

We shall examine the following nonlinear one-dimensional steady-state problem with boundary conditions of the first kind:

$$\frac{d}{dx}\left[a(u)\frac{du}{dx} + a(u)\delta(t)\frac{dt}{ax}\right] = h(u), \tag{2}$$

$$\frac{d}{dx}\left[\lambda(t)\frac{dt}{dx}\right] = 0, \tag{3}$$

$$x = 0,\ t = t_1 = \text{const};\ x = R,\ t = t_2 = \text{const};$$
$$x = 0,\ u = u_1 = \text{const};\ x = R,\ u = u_2 = \text{const}. \tag{4}$$

Equation (3) can be fully solved for quite a wide class of functions $\lambda(t)$ and various boundary conditions. After determining the form of function $t = t(x)$, we can represent the derivative $\delta(t)dt/dx$, in (2) as some known function $f_1(x)$.

When $h \neq 0$, Eq. (2) transforms to a nonlinear equation of the type

$$a(u)\frac{d^2u}{dx^2} + a'(u)\left(\frac{du}{dx}\right)^2 + a'(u) f_1(x)\frac{du}{dx} + a(u) f_1'(x) = h(u), \tag{5}$$

where $a'(u)$ and $f_1'(x)$ denote differentiation with respect to u and x, respectively.

If in (2) $h(u) = 0$, we obtain instead of (5) the simpler nonlinear equation

$$a(u)\,du/dx + a(u) f_1(x) = C_3, \tag{6}$$

where C_3 is a constant.

Therefore the solution of (2) and (3), both for boundary conditions of the first kind and for other boundary conditions, leads in practice to the solution of nonlinear equations (5) or (6).

If functions $a(u)$, $h(u)$, and $f_1(x)$ are such that a solution to (5) or (6) can be obtained, the problem (2)–(4) has an exact analytical solution.

I. for example, let $a(u) = 1/u$, $h(u) = 0$, $\lambda(t) = \delta(t) = bt$, where b = const. With account for the first two boundary conditions, in this case (3) gives

$$t^2 = (2/b)(C_1 x + C_2), \tag{7}$$

where

$$C_1 = (b/2R)(t_2^2 - t_1^2), \quad C_2 = (b/2)\,t_1^2.$$

Substituting the values of the constants into (7), we finally obtain for function t the expression

$$t^2 = (t_2^2 - t_1^2)\,x/R + t_1^2. \tag{8}$$

We shall determine the function on $u = u(x)$.

After substitution of the expressions for $a(u)$, $\delta(t)$ and dx/dt, Eq. (2) leads to an ordinary linear equation in u

$$du/dx - C_3 u + C_1 = 0.$$

The general solution of this equation is

$$u = C_3^{-1}\,[C_1 - C_4 \exp(C_3 x)].$$

The constant C_1 is determined from the formula given above.

For determining the constants C_3 and C_4, after using the second two boundary conditions we obtain the following set of equations:

$$C_4 = C_1 - C_3 u_1,$$
$$C_4 \exp(C_3 R) = C_1 - C_3 u_2.$$

These equations must satisfy the condition

$$C_3 \neq 0.$$

The latter is the boundedness condition for u.

It is easily verified, moreover, that when $C_3 = 0$, Eq. (2) vanishes.

Constants C_3 and C_4 are determined graphically from the set of equations obtained. For example, if it is assumed that $u_1 = 3$, $u_2 = 1$, $b = R = 1$ and $C_1 = 4$, the values of C_3 and C_4 from the formulas obtained are approximately: $C_3 = 0.92908$, $C_4 = 1.21276$.

II. Now let $a(u) = u$, $h(u) = 0$, $\lambda(t) = bt$, $\delta(t) = \delta_0 t$, where b and δ_0 are constants.

We use the previous expression (8) for function t, and for u we obtain the nonlinear equation

$$u\,du/dx + ku - C_3 = 0, \tag{9}$$

where the constant $k = \dfrac{\delta_0}{2R}(t_2^2 - t_1^2)$.

The general solution of (9) is

$$\frac{u}{k} + \frac{C_3}{k^2}\ln(ku - C_3) = C_4 - x. \tag{10}$$

Using the last two boundary conditions, we obtain equations for C_3 and C_4

$$u_1 + \frac{C_3}{k}\ln(ku_1 - C_3) = kC_4, \tag{11}$$

$$u_2 + \frac{C_3}{k}\ln(ku_2 - C_3) = k(C_4 - R). \tag{12}$$

It is clear from the solution of (10) that the constant C_3 must satisfy the conditions

$$ku_1 - C_3 > 0, \quad ku_2 - C_3 > 0.$$

Eliminating C_4 from (11) and (12), we obtain for C_3 the equation

$$\exp\{k[kR-(u_1-u_2)]/C_3\}=(ku_1-C_3)/(ku_2-C_3), \tag{13}$$

which is solved graphically.

Having determined C_3 from (13), we find C_4 either from (11) or from (12).

As an example, the set of equations (1), (12) has been solved for conditions $u_1 = 3$, $u_2 = 1$, $k = 1$, $R = 1$.

The following approximate values were found:

$$C_3 = -1.91328 \text{ and } C_4 = -0.04549.$$

III. We shall examine the problem when Eq. (2) contains a source depending on u.

Let

$$\lambda(t)=\exp(t),\quad \delta(t,\ u)=\exp(t)/u,\quad a(u)=u,\quad h(u)=-1/2u.$$

Solving (3) with the first two boundary conditions, we obtain for the function t

$$\exp(t)=C_1x+C_2;$$

$$C_1=\frac{1}{R}\{\exp(t_2)-\exp(t_1)\},\quad C_2=\exp(t_1). \tag{14}$$

We shall find $u = u(x)$.

The nonlinear equation (2) becomes

$$u\frac{d^2u}{dx^2}+\left(\frac{du}{dx}\right)^2+\frac{1}{2u}=0. \tag{14'}$$

Equation (14'), as is known, may be written in the form

$$u^2\left(\frac{du}{dx}\right)^2+u=C_3.$$

The general integral of this equation is

$$4C_3^3-3C_3u^2-u^3=\frac{9}{4}(x+C_4)^2. \tag{15}$$

Using (15) and the last two boundary conditions, we find

$$4C_3^3-3C_3u_1^2-u_1^3=\frac{9}{4}C_4^2,$$

$$4C_3^3-3C_3u_2^2-u_2^3=\frac{9}{4}(R+C_4)^2.$$

From these equations the constants C_3 and C_4 are evaluated.

IV. Let $\lambda(t)=\exp(t)$, $a(u)=\exp(-ku)$, $\delta(t,\ u)=\lambda(t)\exp(ku)$, $h(u)=b_1u\exp(-ku)$, $k>0$, $b_1>0$; then Eq. (2) takes the form

$$\frac{d^2u}{dx^2}-k\left(\frac{du}{dx}\right)^2-b_1u=0. \tag{16}$$

Equation (16) occurs in the theory of nonlinear oscillations. By substituting $p(u) = (du/dx)^2$ we can transform it to a linear equation in p:

$$\frac{dp}{du}-2kp-2b_1u=0.$$

From this we find

$$p=C_3\exp(2ku)-(b_1/2k^2)(2ku+1). \tag{17}$$

Denoting the right side of (17), for brevity, by $\Phi^2(u)$, and performing certain calculations, we obtain a relation for the function $u = u(x)$

$$\int du/\Phi(u) = x + C_4. \tag{18}$$

Formula (18) is the general integral of (16). After integration, constants C_3 and C_4 are determined using the last two boundary conditions. The solution of (3) in case IV for $t = t(x)$ is given as before by (14). Thus, in case IV the solution of the nonlinear problem (2)-(4) is determined by (18) and (14). If we take another type of source, namely, put $h(u) = -b_1 \exp(-ku)$, then instead of (16) we obtain the simpler equation

$$\frac{d^2u}{dx^2} - k\left(\frac{du}{dx}\right)^2 + b_1 = 0, \tag{19}$$

whose general integral has the form

$$\exp[2ka_1(x + C_4)] = (M - a_1)/(M - a_1), \tag{20}$$

where

$$M^2(u) = C_3 \exp(2ku) + a_1^2, \quad a_1^2 = b_1/k > 0.$$

Using the last two boundary conditions to evaluate C_3 and C_4, we obtain equations which are solved graphically:

$$\exp[2ka_1C_4] = (M_1 - a_1)/(M_1 + a_1), \tag{21}$$

$$\exp[2ka_1(R + C_4)] = (M_2 - a_1)/(M_2 + a_1), \tag{22}$$

where

$$M_1 = M(u_1), \quad M_2 = M(u_2).$$

Let us take a special case. When $k = 1/2 = b_1$, we have $a_1^2 = 1$. Then (20)-(22) assume the simpler form:

$$\exp(x + C_4) = (N - 1)/(N + 1),$$

$$\exp(C_4) = (N_1 - 1)/(N_1 + 1),$$

$$\exp(R + C_4) = (N_2 - 1)/(N_2 + 1),$$

where

$$N^2(u) = C_3 \exp(u) + 1, \quad N_1 = N(u_1), \quad N_2 = N(u_2).$$

11 January 1965

All-Union Correspondence of the Food Industry
Institute, Moscow